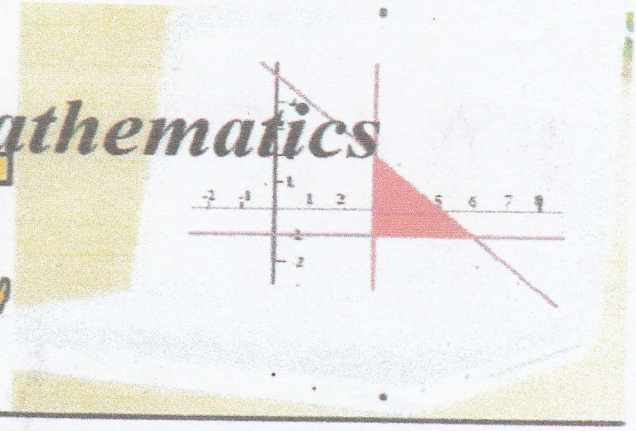
A'level Mathematics

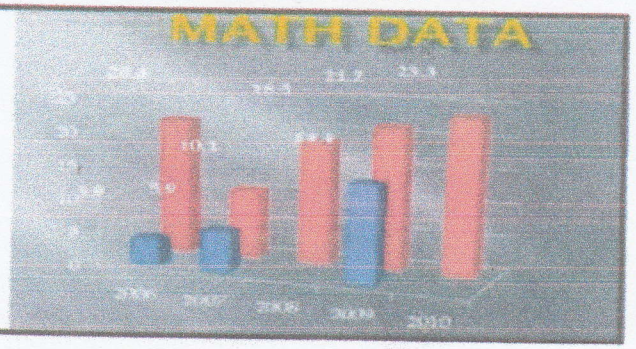


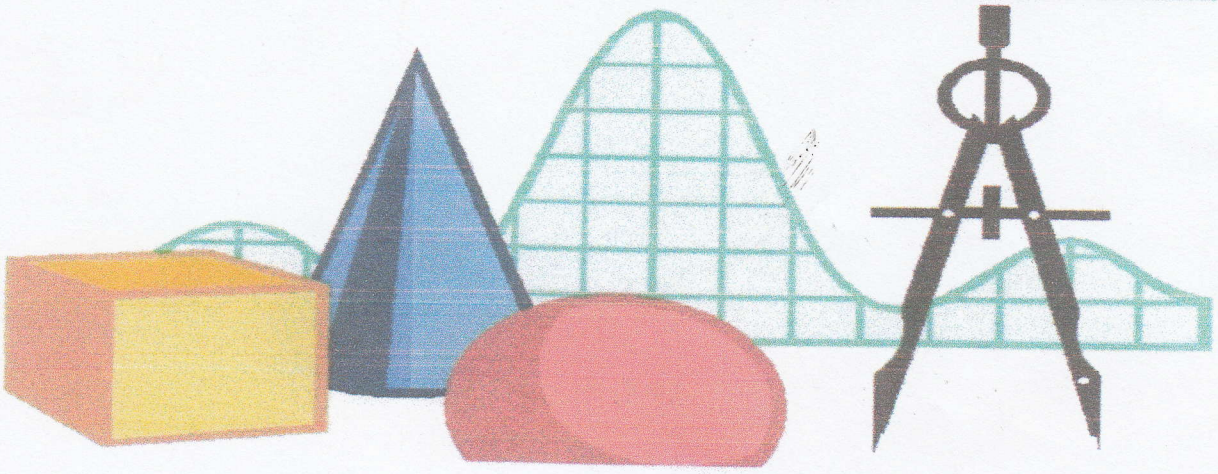
A'LEVEL MATHEMATICS NATIONAL EXAMINATIONS QUESTIONS AND ANSWERS FROM 2005 - 2015

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$$

Example:

$$\int x^5 \, dx = \frac{1}{6} x^6 + c$$





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INTRODUCTION

The document is the compilation of Advanced level Mathematics national examination questions and answers done from 2005 up to 2015 academic year.

The document serves as helping tool for students as well as for teachers. Students will use it when preparing to do their exams.

On the side of teachers, it provides them with exercises, assignments and questions according to the national curriculum. Therefore, this work will contribute remarkably to the success of students from their Advanced level chemistry national exams.

I would wish to take this time to thank my family and all others who helped me in preparation of this document.

Table of content

INTRODUCTION 1
Table of content 2
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2005 4
SECTION A: Attempt all questions. (55 marks) 4
SECTION B: Choose three questions in this section (45 marks each) 9
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2006 13
SECTION A: Attempt all questions. (55 marks) 13
SECTION B: Choose three questions in this section (45 marks each) 18
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2007 23
SECTION A: Attempt all questions. (55 marks) 23
SECTION B: Choose three questions in this section (45 marks each) 30
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2008 36
SECTION A: Attempt all questions. (55 marks) 36
SECTION B: Attempt ANY THREE questions (45 marks) 42
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2009 46
SECTION A: Attempt all questions. (55 marks) 46
SECTION B: ANSWER ANY THREE QUESTIONS (45 marks) 54
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2010 60
SECTION A: Attempt all questions. (55 marks) 60
SECTION B: Attempt any three questions. (45 marks) 66
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2011 71
SECTION A: Attempt all questions. (55 marks) 71
SECTION B: Attempt any three questions. (45 marks) 76
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2012 80
SECTION A: Attempt all questions. (55 marks) 81
SECTION B: Attempt ONLY THREE questions (45 marks) 86
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2013 91
SECTION A: Attempt all questions. (55 marks) 91
SECTION B: Attempt ONLY THREE questions (45 marks) 96
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2014 100
SECTION A: Attempt all questions. (55 marks) 100

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2005
BIOLOGY-CHEMISTRY

SECTION A: Attempt all questions. (55 marks)

01. Solve: log(x+1) + log(x-1)=2log(x-2) 4 marks

Answer:

Solve in $\Re$: log(x+1) + log(x-1) = 2 log(x-2)

Initial conditions:

$\begin{cases} x+1>0 \\ x-1>0 \\ x-2>0 \end{cases} \Leftrightarrow x>2$

log(x+1) + log(x-1) = 2 log(x-2)

$\Leftrightarrow \log(x+1)(x-1) = \log(x-2)^2$

$\Leftrightarrow (x+1)(x-1) = (x-2)^2 \Leftrightarrow x^2-1 = x^2-4x+4$

$\Leftrightarrow 4x = 5 \Leftrightarrow x = \frac{5}{4} \Leftrightarrow x = 1.25$

The equation has no solution in $\Re$ because $\frac{5}{4} < 2$

02. Given that $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, calculate, without using a calculator, the exact value of $\cos\frac{5\pi}{12}$ and $\sin\frac{5\pi}{12}$ 4 marks

Answer:

a) $\text{Cos}\left(\frac{2\pi}{12}\right) = \text{Cos}\left(\pi - \left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right) = -\left(cos\frac{\pi}{4}cos\frac{\pi}{3} - sin\frac{\pi}{4}sin\frac{\pi}{3}\right) = -\left(\frac{\sqrt{2}}{2}.\frac{1}{2} - \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

b) $\text{Sin}\left(\frac{5\pi}{12}\right) = \sin\left(\pi - \frac{7\pi}{12}\right) = \sin\left(\pi - \left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right) = sin\frac{\pi}{4}cos\frac{\pi}{3} + cos\frac{\pi}{4}sin\frac{\pi}{3} = \left(\frac{\sqrt{2}}{2}.\frac{1}{2} + \frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

03. The lengths of sides of a triangle are 7m, 10m and 16m. Calculate the size of each angle of the triangle. Give the answer correct to the nearest degree. 5 marks

Answer:

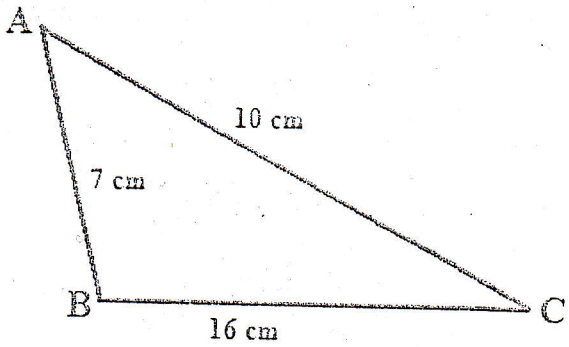
The measures associated to the irregular triangle are proportional

$\text{CosA} = \frac{-256+49+100}{2x10x7} = \frac{-107}{140} \Leftrightarrow A = \cos^{-1}\left(\frac{-107}{140}\right) \Leftrightarrow A = 140^0$

$\text{Cos B} = \frac{-256+49-100}{2x16x7} = \frac{205}{224} \Leftrightarrow B = \cos^{-1}\left(\frac{205}{224}\right) \Leftrightarrow B = 24^0$

$\text{Cos C} = \frac{256-49+100}{2x10x16} = \frac{307}{320} \Leftrightarrow C = \cos^{-1}\left(\frac{307}{320}\right) \Leftrightarrow C = 16^0$

SECTION B: ATTEMPT THREE QUESTIONS ONLY (45 marks) 107
ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2015 114
SECTION B: ATTEMPT THREE QUESTIONS ONLY. (45 marks) 120



04. You are given the system of equations:

$\begin{cases} kx - 9y = -3 \\ 4x + (k-12)y = k \end{cases}$, where k is a real number.

For which value of k does the system of equations have no solution? **4 marks**

Answer:

$\begin{cases} kx - 9y = -3 \\ 4x + (k-12)y = k \end{cases}$

The system has no solution in $\Re x \Re$ if and only if

$\begin{vmatrix} k & -9 \\ 4 & k-12 \end{vmatrix} = 0 \Leftrightarrow k^2\text{-}12k+36 = 0 \Leftrightarrow k = 6$

$\begin{vmatrix} -3 & -9 \\ k & k-12 \end{vmatrix} \neq 0 \Leftrightarrow 6k+36 \neq 0 \Leftrightarrow k \neq -6$

$\begin{vmatrix} k & -3 \\ 4 & k \end{vmatrix} \neq 0 \Leftrightarrow k^2 + 12 \neq 0$ and $k^2+12 > 0$

Hence the system has no solution for k = 6

05. Consider the function f defined by $f(x)=\frac{x^2-4x+3}{x-3}$. Is the function f continuous in the set of real numbers IR? If not, redefine it so that it is continuous. **2 marks**

Answer:

$f(x)=\frac{x^2-4x+3}{x-3}$

The function is not continuous, it is discontinuous at x = 3

The function can be redefined by:

$$\begin{cases} \frac{x^2-4x+3}{x-3} & if\ x \neq 3 \\ \frac{0}{0} \ldots\ldots & if\ x = 3 \end{cases}$$

06. Given the function f: IR → IR: $x \rightarrow \log_x \sqrt{1-x^2}$, determine the values of x so that the function f is defined. **2 marks**

Answer:

$f: \Re \rightarrow \Re: x \rightarrow \log_x \sqrt{1-x^2}$

The function f is defined for $\begin{cases} 1-x^2 > 0 \\ x > 0 \text{ and } x \neq 1 \end{cases} \Leftrightarrow \begin{cases} 1-x^2 > 0 \\ x > 0 \text{ and } x \neq 1 \end{cases} \Leftrightarrow$ **for**

$\begin{cases} -1 < x < 1 \\ x > 0 \text{ and } x \neq 1 \end{cases}$

The function is defined for $0 < x < 1$

07. The following are percentages of milk fats from 20 samples collected from the milk brought by famers to a milk collecting centre:
4.12 4.04 3.96 3.95 3.98 3.94 3.98 4.12 3.98 3.96
3.95 4.02 3.95 4.02 4.04 3.98 4.02 4.02 3.95 4.02
Calculate the standard deviation of the data. **4 marks**

Answer:

To calculate the standard deviation of those numbers:
- **Work out the Mean (the simple average of the numbers)**
- **Then for each number: subtract the Mean and square the result.**
- **Then work out the mean of those squared differences (variance V(x)**
- **Take the square root of that and we are done (σ_x)**

x_i	f_i	$x_i f_i$	$(x_i - \bar{x})^2 f_i$
3.94	1	3.94	$(-0.06)^2 \times 1 = 0.0036$
3.95	4	15.8	$(-0.05)^2 \times 4 = 0.01$
3.96	2	7.92	$(-0.04)^2 \times 2 = 0.0032$
3.98	4	15.92	$(-0.02)^2 \times 4 = 0.0016$
4.02	5	20.1	$(0.02)^2 \times 5 = 0.002$
4.04	2	8.08	$(0.04)^2 \times 2 = 0.0032$
4.12	2	8.24	$(0.12)^2 \times 2 = 0.0288$
	20	$\sum x_i f_i = 80$	$\Sigma(x_i - \bar{x})^2 f_i = 0.0524$

$$\text{Mean} = \frac{80}{20} = 4$$

$$V(x) = \frac{0.0524}{20} = 0.00262$$

$$\sigma_x = \sqrt{0.00262} = 0.05118$$

08. The equation of the curve is $y = x + \frac{4}{x}$.

a) Write the equations of the asymptotes of the curve. **1 mark**
b) Find the maximum and minimum value of y. **2 marks**
c) Sketch the graph of the curve. **2 marks**

Answer:

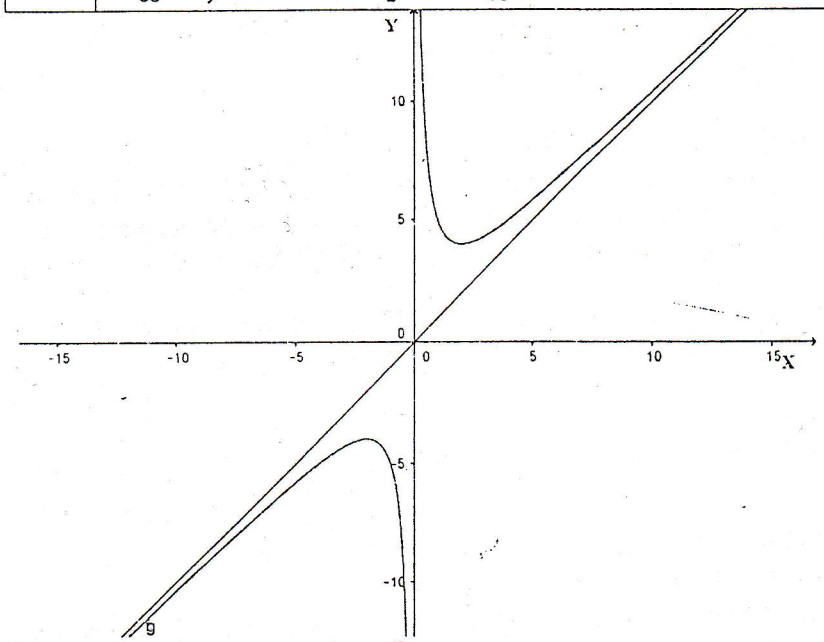
a) Vertical asymptote: VA $\equiv x = 0$
Horizontal asymptote: HA $\equiv y = 0$

b) $y' = 1 - \frac{4}{x^2}$
$y' = 0 \Leftrightarrow x = 2$ or $x = -2$

c)

X	$-\infty$		-2		0		+2		$+\infty$
f'(x)	+	+ + + +	0	- - - - - - -	II	- - - - - -	0	+ + + + +	+
f(x)	$+\infty$		↘		$+\infty$		↗		$+\infty$
			-4				+4		

	$-\infty$ ↗	↘ $-\infty$



09. Given the complex number $z = \frac{3-i\sqrt{3}}{3+i\sqrt{3}}$, write it in trigonometric from and write z^{18} in algebraic form. **5 marks**

Answer:

The complex number $Z = \frac{3-i\sqrt{3}}{3+i\sqrt{3}}$

$Z = \left(\frac{3-i\sqrt{3}}{3+i\sqrt{3}}\right)\left(\frac{3-i\sqrt{3}}{3-i\sqrt{3}}\right) = \frac{6-6i\sqrt{3}}{12} = \frac{1}{2} - \frac{6i\sqrt{3}}{2} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$

$|Z| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

The argument of Z is determined by

$$\begin{cases} cos\theta = \frac{1}{2} \\ sin\theta = -\frac{\sqrt{3}}{2} \end{cases} \Leftrightarrow \theta = -\frac{\pi}{3}$$

Trigonometric form of Z is

$Z = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

So

$Z^{18} = \left(cos\frac{\pi}{3} - isin\frac{\pi}{3}\right)^{18} = cos\frac{18\pi}{3} - isin\frac{18\pi}{3} = \cos 6\pi - i\sin 6\pi = 1$

Algebraic form of $Z^{18} = 1$

10. Utilising Moivre's theorem, express sin5x as a polynome in sinx. **3 marks**

Answer:

$cos5x + isin5x = (cosx + sinx)^5 = \cos^5 x + 5\cos^4 x\,(i\sin x) + 10\cos^3 x(i\sin x)^2 + 10\cos^2 x(i\sin x)^3 + 5\cos x\,(i\sin x)^4 + (i\sin x)^5 =$

$cos^5x - 10cos^3xsin^2x + 5cosxsin^4x + i(5cos^4xsinx - 10cos^2xsin^3x + sin^5x)$

$$sin5x = 5(1-sin^2x)^2sinx - 10(1-sin^2x)sin^3x + sin^5x$$

$$\Leftrightarrow sin5x = 16sin^5x - 20sin^3x + 5sinx.$$

11. Calculate: $\int_0^{\pi/4}(cos^2x - sin^2x)dx$. 3 marks

Answer:

$\int_0^{\pi/4}(cos^2x - sin^2x)dx = \int_0^{\pi/4} cos2xdx = \frac{1}{2}[sin2x]_0^{\frac{\pi}{4}} = \frac{1}{2}$

12. Write the equation of ellipse E which is in a Cartesian plane (o, i, j). The centre of the ellipse is the origin, the focal points of the ellipse are on y-axis and the major axis is along y-axis. The height of the large axis of the ellipse is 10 and the distance between the focal points is 8. 2 marks

Find the coordinates of the vertices and the coordinates of the focal points of the ellipse. 3 marks

Answer:

Semi major axis a = 5

Semi minor axis b = $\sqrt{25-16} = 3$

$E \equiv \frac{x^2}{9} + \frac{y^2}{25} = 1$

The vertices of ellipse are: $V_1(3, 0)$ $V_2(-3, 0)$ $V_3(0, 5)$ $V_4(0, -5)$

The foci of the ellipse are $F_1 = (0, 4)$, $F_2 = (0, -4)$

13. In a vectorial space E_o with a unit base and perpendicular axes $(\vec{i}, \vec{j}, \vec{k})$, the points a, b andc are such that $\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} - \vec{k}$ and $\vec{c} = 2\vec{j} + \vec{k}$. Find points P in plane (oab) if $\vec{p}$ is a unit vector perpendicular to $\vec{c}$. 5 marks

Answer:

Let $\overrightarrow{p}(x, y, z), \overrightarrow{a}, \overrightarrow{b}$ *and* $\vec{p}$ *are copular*

$\Leftrightarrow (\vec{a} \wedge \vec{b}).\vec{p} = 0$

$\Leftrightarrow \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} x + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} y + \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} z = 0$

$\Leftrightarrow x + 5y + 2z = 0$

$\vec{p}$ is a unit vector, hence $\|\vec{p}\| = \sqrt{x^2 + y^2 + z^2} = 1$

$\vec{p}$ is perpendicular to $\vec{c} \Leftrightarrow \vec{p}.\vec{c} = 0 \Leftrightarrow 2y+z = 0 \Leftrightarrow z = -2y$

and $\begin{cases} x + y = 0 \\ x^2 + 5y^2 = 1 \end{cases} \Leftrightarrow y = -x$ which gives $x^2+5x^2 = 1$ $\begin{cases} x = \pm\sqrt{\frac{1}{6}} \\ y = \pm\sqrt{\frac{1}{6}} \\ z = \pm 2\sqrt{\frac{1}{6}} \end{cases}$

The points of p are $\pm\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{6}}, 2\sqrt{\frac{1}{6}}\right)$

14. A national commission of the young artisans is composed of 7 girls and 5 boys. 4 delegates chosen at random from the commission have an audience with MINICOM. What is the probability that at least on girl is chosen? What you conclude about the event of at least one girl being chosen? 4 marks

Answer:

The possible outcomes $C_7^1xC_7^3 + C_7^2xC_5^2 + C_7^3xC_5^1 + C_7^4$

P(at least one girl) $= \frac{C_7^1 x C_7^3 + C_7^2 x C_5^2 + C_7^3 x C_5^1 + C_7^4}{C_{12}^4} = 0.99$

The probability is approximately equal to 1, the event is certain to choose at least one girl.

SECTION B: Choose three questions in this section (45 marks each)

15. The function f is defined by $f(x)=\frac{\sqrt{1-x^2}}{x}$.

a) Determine the definition of the domain of the function f and the limits of the boundaries of the domain. 4 marks

b) Write the equations of asymptotes if available. 1 mark

c) Determine the first derivative and the second derivative of the function f. 5 marks

d) Determine the variations of the function f, the points of inflexion and the nature of the curve representing the function f. 3 marks

e) Sketch the graph of the curve of the function f. 2 marks

Answer:

a) The function is defined for

$f(x) = \frac{\sqrt{1-x^2}}{x}$

$\begin{cases} 1 - x^2 \geq 0 \\ x \neq 0 \end{cases}$

$Df = [-1, 0[\cup]0, 1]$

The limits of the boundaries of the domain

$\lim_{x\to -1} f(x) = 0$

$\lim_{x\to 1} f(x) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|\sqrt{\frac{1}{x^2}-1}}{x} = \left(-\sqrt{\frac{1}{x^2}-1}\right) = -\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|\sqrt{\frac{1}{x^2}-1}}{x} = \left(\sqrt{\frac{1}{x^2}-1}\right) = +\infty$$

b) The vertical asymptote VA≡x = 0

c) $f'(x) = \left(\frac{\sqrt{1-x^2}}{x}\right)' = \frac{(1-x^2)^{\frac{1}{2}\prime} x - (1-x^2)^{\frac{1}{2}}.1}{x^2} = \frac{\frac{1}{2}(-2x)x(1-x^2)^{-\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}.1}{x^2} =$

$\frac{-x^2(1-x^2)^{-\frac{1}{2}} - (1-x^2)^{\frac{1}{2}}.1}{x^2}$

$= \frac{-1}{(1-x^2)^{\frac{1}{2}}} - \frac{(1-x^2)^{\frac{1}{2}}}{x^2} = \frac{-x^2-(1-x^2)}{x^2(1-x^2)^{\frac{1}{2}}} = -\frac{1}{x^2(1-x^2)^{\frac{1}{2}}}$

$f'(x) = -\frac{1}{x^2(1-x^2)^{\frac{1}{2}}}$

$f''(x) = \left(-\frac{1}{x^2(1-x^2)^{\frac{1}{2}}}\right)' = \frac{0-(-1)\left[x^2(1-x^2)^{\frac{1}{2}}\right]'}{x^4(1-x^2)} = \frac{-\left[-(1)(2x)(1-x^2)^{\frac{1}{2}} + x^2(-\frac{2x}{2})(1-x^2)^{-\frac{1}{2}}\right]}{x^4(1-x^2)} =$

$\frac{+2x(1-x^2)^{\frac{1}{2}}}{x^3(1-x^2)} + \frac{(-x^2)}{x^3(1-x^2)} = \frac{2(1-x^2)-x^2}{x^3(1-x^2)^{3/2}} = \frac{2-3x^2}{x^3(1-x^2)^{3/2}}$

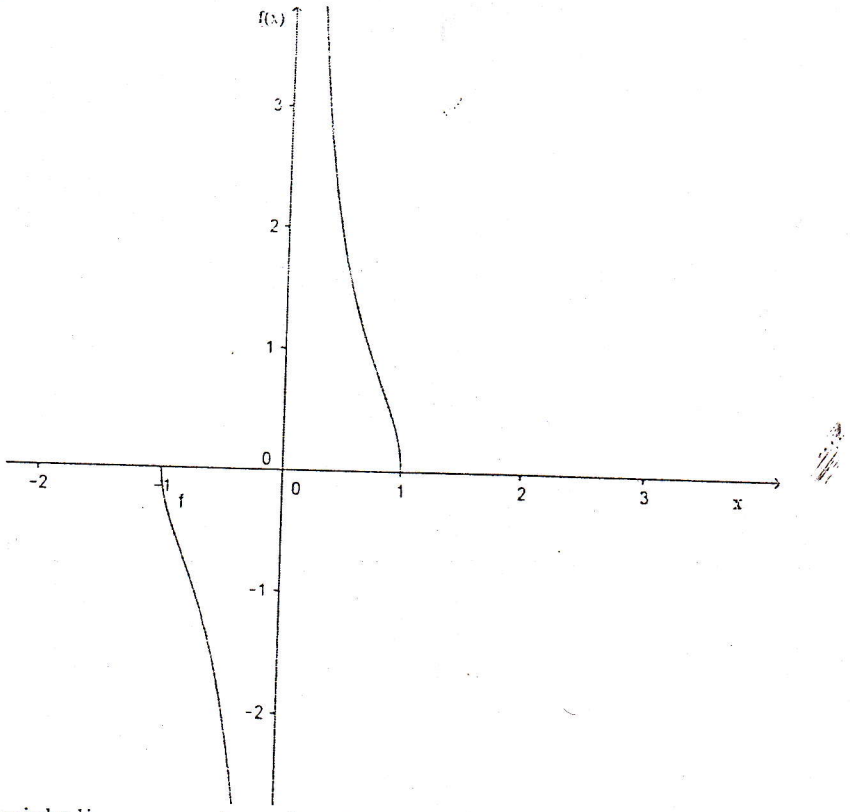
d)

X	-1	$-\sqrt{2}/3$	0	$\sqrt{2}/3$	+1
f'(x)	- - - - - - - - - -		- - - - - - - - - -		- - - - - - - - - -
f''(x)	++++++++++	0	----------	++++++++++	0 ----------
f(x)	0	$-\sqrt{3}/2$	$+\infty$ / $-\infty$	$\sqrt{3}/2$	0

Points of inflection $(-\frac{\sqrt{2}}{3}, -\frac{\sqrt{3}}{2})$ and $(\frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{2})$

Nature of curve is symmetrical with respect to origin (it is odd function)

e)



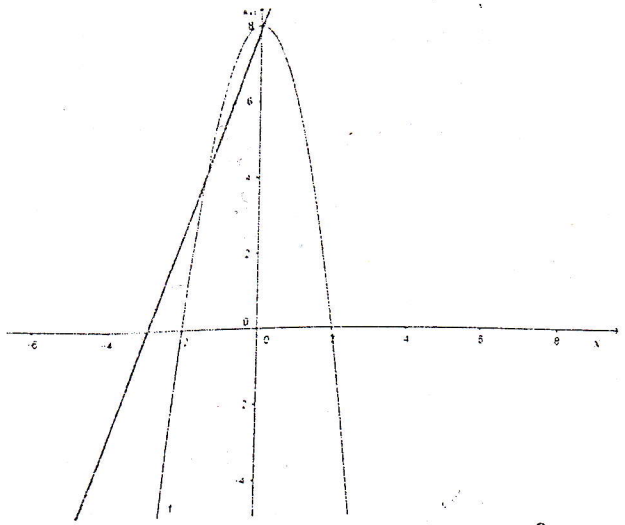
16. A straight line passes through points A(-1, 5) and B(0, 8) and $y=8-2x^2$ is the equation of the curve C.
    a) Find the equation of the straight line AB. **2 marks**
    b) In the same Cartesian plane draw the straight line AB and the curve C. **3 marks**
    c) Calculate the area between the curve C and the straight line AB. **6 marks**
    d) Calculate the volume of solid of revolution about the x-axis of the surface area in 16(c). **4 marks**

Answer:

a) The equation of the straight line AB is

$y-5=\frac{8-5}{0+1}(x+1) \Rightarrow y=3x+8$

b)



c) $8-2x^2=3x+8 \Rightarrow x(2x+3)=0 \Rightarrow x=0$ or $x=\frac{-3}{2}$

The area $A=\int_{-3/2}^{0}[(8-2x^2)-(3x+8)]dx=\int_{-3/2}^{0}[(-2x^2-3x)]dx=\left[\frac{-2x^3}{3}-\frac{3x^2}{2}\right]_{-3/2}^{0}=\frac{9}{8}$ UA

d) The volume of solid of revolution $V=\pi\int_{-3/2}^{0}(2x^2-3x)^2dx-\pi\int_{-3/2}^{0}(3x+8)^2dx$

=

$\pi\left[64x-\frac{32x^3}{3}+\frac{4x^5}{5}\right]_{-3/2}^{0}-\pi\left[\frac{9x^3}{3}+\frac{24x^2}{2}+64x\right]_{-3/2}^{0}=\frac{2643\pi}{40}-\frac{417\pi}{8}=\frac{558\pi}{40}$UV

17. Each seat of Gacaca judiciary court of a cell is made up with nine (9) persons of integrity.

a) How many ways of choosing a committee of coordinators of judiciary if each member of Gacaca seat is a candidate and electrons are made in three successive phases for the posts of: 1. President; 2. First and second presidents; 3. Two secretaries? **6 marks**

b) If the seat of Gacaca court of the cell is composed of 4 ladies:

i) What is the probability of choosing a lady as the president of the judiciary court? 1 mark

ii) What is the probability for the two posts of first and second vice presidents to be occupied by ladies? **5.5 marks**

iii) What is the probability of choosing two secretaries of different sexes if the president and one of the vice presidents are men? **2.5 marks**

N.B: For every question the five deputies of persons of integrity are not included.

Answer:

a) The number of ways of choosing a committee of coordinators = $C_9^1 x A_3^2 x C_6^2 = 7560$

b) i) The probability of choosing a lady as the president of the judiciary court is $\frac{4}{9}$

ii) The probability for the two posts of first and secondary vice- presidents to be occupied by ladies $= \frac{A_4^2+A_3^2}{A_8^2} = \frac{18}{56} = \frac{9}{28}$

iii) The probability of choosing two secretaries of different sexes if the president and one of the vice-presidents are men is $\frac{3x6}{C_6^2} = \frac{9}{15} = \frac{3}{5}$

18. The vectorial space E_o is a unit base with perpendicular axes $(\vec{i}, \vec{j}, \vec{k})$ and you are given vectors of E_o such that: $\vec{u} = \begin{pmatrix}2\\3\\6\end{pmatrix}, \vec{v} = \begin{pmatrix}2\\-6\\3\end{pmatrix}, \vec{w} = \begin{pmatrix}3\\3\\6\end{pmatrix}$.

a) Show that $(\vec{u}, \vec{v}, \vec{w})$ is a base of E_o. 4 marks

b) Determine the coordinates of the vectors $\vec{e} = \vec{i} - \vec{j} + \vec{k}$ in the base $(\vec{u}, \vec{v}, \vec{w})$. 6 marks

c) Write parametric equations and Cartesian equations of the straight line uv. 5 marks

Answer:

a) $(\vec{u}, \vec{v}, \vec{w})$ is the basis of E_o if

$(\vec{u} \wedge \vec{v}).\vec{w}) \neq 0$

$\vec{u} \wedge \vec{v} = \left(\begin{vmatrix}3 & -6\\6 & 3\end{vmatrix} - \begin{vmatrix}2 & 2\\6 & 3\end{vmatrix}, \begin{vmatrix}2 & 2\\3 & -6\end{vmatrix}\right) = (45, 6, -18)$

$(\vec{u} \wedge \vec{v}).\vec{w}) = (45, 6, -18).(3, 3, 6) = 135 + 18 - 108 = 45 \neq 0$

$\Leftrightarrow (\vec{u} \wedge \vec{v}).\vec{w})$ *is the basis* of E_o.

b) $\vec{e} \in E_o \; \exists (a, b, c) \in \mathbb{R}^3$

$\vec{e} = \vec{i} - \vec{j} + \vec{k} = a\vec{u} + b\vec{v} + c\vec{w} = a(2\vec{i} + 3\vec{j} + 6\vec{k}) + b(2\vec{i} - 6\vec{j} + \vec{k}) + c(3\vec{i} + 3\vec{j} + 6\vec{k}) = (2a+2b+3c)\vec{i} + (3a-6b+3c)\vec{j} + (6a+3b+3c)\vec{k} \Leftrightarrow \begin{cases} 2a + 2b + 3c = 1 \\ 3a - 6b + 3c = -1 \\ 6a + 3b + 6c = 1 \end{cases}$

$\Delta = \begin{vmatrix}2 & 2 & 3\\3 & -6 & 3\\6 & 3 & 6\end{vmatrix} = -90 + 135 - 0 = 45$

$\Delta_a = \begin{vmatrix}1 & 2 & 3\\-1 & -6 & 3\\1 & 3 & 6\end{vmatrix} = -18$

$\Delta_b = \begin{vmatrix}2 & 1 & 3\\3 & -1 & 3\\6 & 1 & 6\end{vmatrix} = 9$

$\Delta_c = \begin{vmatrix}2 & 1 & 3\\3 & -1 & 3\\6 & 1 & 6\end{vmatrix} = 21$

$a = \frac{-18}{45}, b = \frac{9}{45} = \frac{1}{5}, c = \frac{21}{45} = \frac{7}{15}$

c) The direction vector of the line is (0, 9, 3), parametric equations of the line uv are

$\begin{cases} x = 2 \\ y = -6 + 9t \\ z = 3 + 3t \end{cases}$

Cartesian equations are

$\begin{cases} y = -6 + 9t \\ z = 3 + 3t \end{cases} \Leftrightarrow t = \frac{y+6}{9} = \frac{z-3}{3}$

$\Leftrightarrow y - 3z = -15$

19. With aid of complex numbers:

a) Solve the equation $\sqrt{3}cosx - sinx + \sqrt{3}=0$ 9 marks

b) Expand cos^6x and calculate $\int_{-\pi}^{0} cos^6x dx$ 6 marks

Answer:

a) $\sqrt{3}cosx - sinx + \sqrt{3} = 0$

$z = a+bi = \sqrt{3} - i$

$|z| = 2$

$\begin{cases} cos\theta = \frac{\sqrt{3}}{2} \\ sin\theta = \frac{-1}{2} \end{cases} \Leftrightarrow \theta = \frac{-\pi}{3}$

$\sqrt{3}cosx - sinx = -\sqrt{3}$

$2cos(x+\frac{\pi}{3}) = -\sqrt{3}$

$cos(x+\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$

$cos(x+\frac{\pi}{6}) = cos\frac{5\pi}{6}$

$x+\frac{\pi}{6} = \pm\frac{5\pi}{6}+2k\pi$

$x = \frac{2\pi}{3}+2\,k\pi$

or

$x = \pi+2\,k\pi$

b) $cos^6x = \left[\frac{1}{2}(z+\frac{1}{z})\right]^6$

$= \frac{1}{2^6}\left(z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\right)$

$= \frac{1}{2^6}\left((z^6+\frac{1}{z^6}) + 6(z^4+\frac{1}{z^4}) + 15(z^2+\frac{1}{z^2}) + 20\right)$

$= \frac{1}{2^6}(cos6x + 6cos4x + 15cos2x + 10)$

$\int_{-\pi}^{0} cos^6x dx = \int_{-\pi}^{0} \frac{1}{2^6}(cos6x + 6cos4x + 15cos2x + 10)dx = \frac{1}{2^5}(\frac{sin6x}{6} + \frac{6sin4x}{4} + 15\frac{sin2x}{2} + 10x)_{-\pi}^{0} = \frac{1}{32}(10\pi) = \frac{5\pi}{16}$

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2006
BIOLOGY-CHEMISTRY

SECTION A: Attempt all questions. (55 marks)

01. Find the quadratic real function $f(x)=ax^2+bx+c$ whose graph passes through the points (5, 0) and has (1, 2) as vertex. 4 marks

Answer:

Let $f(x)=ax^2+bx+c$ and the point on the graph $25a+5b+c = 0$

The vertex is $\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = (1, 2)$

We get $\begin{cases} 25a+5b+c=0 \\ \frac{-b}{2a}=1 \\ \frac{-b^2+4ac}{4a}=2 \end{cases}$

Substitute the value of b by -2a

We get $\begin{cases} 25a+5b+c=0 \\ b=-2a \\ -a+c=2 \end{cases} \Leftrightarrow \begin{cases} a=-\frac{1}{8} \\ b=\frac{1}{4} \\ c=\frac{15}{8} \end{cases}$

Since $f(x)=-\frac{1}{8}x^2+\frac{1}{4}x+\frac{15}{8}$

02. Find the matrix X such that $2X+3A=B$ if

$A=\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$ 3 marks

Answer:

$2X+3A=B$

$\Rightarrow X=\frac{1}{2}(B-3A)$

$X=\frac{1}{2}\left(\begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}-3\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}\right)=\frac{1}{2}\left(\begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}-\begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix}\right)$

$X=\frac{1}{2}\begin{bmatrix} 6 & 6 \\ -1 & 0 \\ -13 & 11 \end{bmatrix}=\begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix} \Rightarrow X=\begin{bmatrix} 3 & 3 \\ -\frac{1}{2} & 0 \\ -\frac{13}{2} & \frac{11}{2} \end{bmatrix}$

03. A company plans to send three of its fifteen managers to a development seminar. How many different groups of three can be sent? 2 marks

Answer:

The number of groups of three can be sent:

$C_{15}^3 or \binom{3}{15}=\frac{15!}{3!12!}=\frac{15x14x13}{2x3}=455$

04. a) Find the coefficient of x^3y^{13} in the expansion of the binomial $(4x-y)^{16}$. 2 marks

b) Find $p(A \cap B)$ if $p(A)=\frac{1}{2}$, $p(B)=\frac{2}{3}$, and $p(A \cup B)=\frac{5}{6}$ where A and B are two events on the same space. 2 marks

Answer:

a) $(4x-y)^6=\sum_{k=0}^{16}(4x)^{16-k}(-y)^k$

The coefficient of x^3y^{13} is $\binom{13}{16}(-1)^{13}=-4\binom{13}{16}=\frac{-4x16!}{13!x3!}=\frac{-4x16x15x14}{3x2}=$

$-4x15x5x7=-2240$

b) $P(A \cap B)=P(A)+P(B)-P(A \cup B)=\frac{1}{2}+\frac{2}{3}-\frac{5}{6}=\frac{3+4-5}{6}=\frac{1}{3}$

05. For the collection of ten numbers:

35 18 19 20 28 18 19 25 35 40,

Find the median, mode and the mean. 5 marks

Answer:

The correction: 18 18 19 19 20 25 28 35 35 40

The median is $\frac{20+25}{2} = 22.5$

The mode are 18, 19 and 35

The mean = $\frac{18x2+19x2+20+25+28+35x2+40}{10} = 25.7$

06. For the given uniform probability density function

$f(x) = \begin{cases} \frac{1}{45} \text{ if } 45 \leq x \leq 90 \\ 0 \text{ elsewhere} \end{cases}$. find the probability that a number chosen at random will be between 50 and 60. 2 marks

Answer:

$P(50 \leq x < 60) = \int_{50}^{60} \frac{1}{45} dx = \left[\frac{1}{45}x\right]_{50}^{60} = \left[\frac{60}{45} - \frac{50}{45}\right] = \left[\frac{10}{45}\right] = \frac{2}{9}$

07. Find the domain and the derivative of the real function $f(x) = \ln(4-2x-x^2)$. 4 marks

Answer:

$f(x) = \ln(4-2x-x^2)$

$4-2x-x^2 > 0 \Rightarrow \{x \in \Re: 4 - 2x - x^2 > 0\}$

$4-2x-x^2 = 5-(x+1)^2$

$(\sqrt{5} - x - 1)(\sqrt{5} + x + 1)$

$4-2x-x^2 > 0 \Rightarrow x \in]-\sqrt{5} - 1, \sqrt{5} - 1[$

Hence domf = $]-\sqrt{5} - 1, \sqrt{5} - 1[$

The derivative of f is $f'(x) \frac{-2x-2}{4-2x-x^2} = \frac{2x+2}{x^2+2x-4}$

08. Let $f(x) = |x + 2| + |-x - 2|$ where $x \in IR$.

a) Find a piecewise definition of f(x) that does not involve the absolute value function. Sketch the graph of *f*. **3 marks**

b) Give the domain and the a range of *f*. **1 mark**

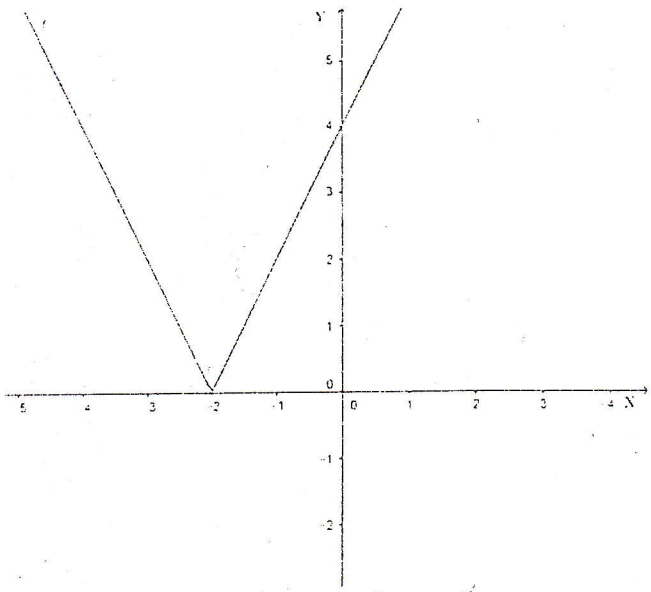
c) Does f' (2) (the derivative of f at x=2) exist? **2 marks**

Answer:

a) $f(x) = |x + 2| + |-x - 2|$

As $|x + 2| = |-x - 2|$, $f(x) = 2(x+2)$

Then $f(x) = \begin{cases} 2(x+2) \text{ if } x \geq -2 \\ -2(x+2) \text{ if } x \leq -2 \end{cases}$



b) $Domf = \Re$ *and the range is* $[0, +\infty[$

c) f is not derivative at $x = 2$ because $f'(2) = 2$ and $f'(2) = -2$

09. In Euclidian space IR^3 the sphere with M(2, -1, 3) as center passes through the point T(1. 2, -3). Write the equation of the sphere and parametric equations of a line which is tangent through T. 4 marks

Answer:

The equation of sphere is $(x-2)^2+(y+1)^2+(z-3)^2 = R^2$

As T(1, 2, -3) is the point of sphere then $(1-2)^2(2+1)^2+(-3-3)^2 = R^2 \Rightarrow R^2 = 46$

Since the equation of sphere is:

$(x-2)^2+(y+1)^2 + (z-3)^2=46$

Or $x^2+y^2+z^2-4x+2y-6z-32 = 0$

Where $d = x_0^2 + y_o^2 + z_0^2 - R^2 = 4+1+9-46 = -32$

$d = -32$

Let $u(\alpha; \beta; \gamma)$ is director external of tangent to the sphere at the point T, you may get

$\vec{u} \perp MT$ *or* $\vec{u}.\overrightarrow{MT} = 0$ *because they are perpendicular then* $-\alpha + 3\beta - 6\gamma = 0$

This solution has many solutions fixed β *and* γ then $\alpha = 3\beta - 6\gamma$

Parametric equations of a line are

$$\begin{cases} x - 1 = (3\beta - 6\gamma)t \\ y - 2 = \beta t \\ z + 3 = \gamma t \end{cases}$$ where t is parameter

10. Let * be a binary operation on the set of integers (Z) by a*b = ab-2 for a, b$\in$ Z. Evaluate (-5)*[6*8] and [(-5)*6]*8. 1 mark

Determine whether * is commutative and whether it is associative. 3 marks

Answer:

Commutativity: a*b = b*a

a*b = ab-2

b*a = ba-2 = ab-2

So it is * is commutative because a*b = b*a

Associativity: (a*b)*c = a*(b*c)

a*b = ab-2

(a*b)*c = (ab-2)*c = (ab-2)c-2 = abc -2c - 2

a*(b*c) = a*(bc-2) = a(bc-2)-2 = abc – 2a-2

⇔ (a*b)*c = a*(b*c) ⇔ abc -2c - 2≠ abc – 2a-2 , thus * is not associative

Evaluate:

(-5)*[6*8]

a*b = ab-2

6*8 = 6.8-2 = 46

(-5)*[6*8] = (-5)*46 = -5x46-2 = -232

(-5)*[6*8] = -232

[(-5)*6]*8 = (-5.6-2)*8 = -32*8 = -32x8 – 2 = -258

(-5)*[6*8] = -258

11. Solve the following inequality where x is real:

$\log_{\frac{1}{2}}(x+1) - \log_{\frac{1}{2}}(x-1) \leq 1$. 3 marks

Answer:

$\log_{\frac{1}{2}}(x+1) - \log_{\frac{1}{2}}(x-1) \leq 1$

$\begin{cases} x+1>0 \\ x-1>0 \end{cases} \Leftrightarrow \begin{cases} x>-1 \\ x>1 \end{cases}$

S =]1, +∞[

$\log_{\frac{1}{2}}(x+1) - \log_{\frac{1}{2}}(x-1) \leq 1 \Leftrightarrow \log_{\frac{1}{2}}(x+1) - \log_{\frac{1}{2}}(x-1) \leq log_{\frac{1}{2}}\frac{1}{2}$

$\frac{(x+1)}{(x-1)} \leq \frac{1}{2} \Leftrightarrow$ 2x+2 ≤ x – 1 ⇔ 2x – x ≤ -1 -2 ⇔ x ≤ -3 ⇔ $]-\infty,-3]$

$S =]-\infty,-3] \cap]1,+\infty[= \emptyset$

12. a) Let f(x) = $3x^2$-5x+7 be a function with real variable x. Find and simplify $\frac{f(x+h)-f(x)}{h}$ for any h and then $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

What is this limit called? 3 marks

b) Evaluate $\lim_{x\to 0}\frac{\ln(1+x)}{\sqrt{x}}$. 2 marks

Answer:

a) f(x) = $3x^2$-5x+7

$\frac{f(x+h)-f(x)}{h} = \frac{[3(x+h)^2-5(x+h)-7]-(3x^2-5x-7)}{h} = \frac{6xh+3h^2-5h}{h} = 6x+3h-5$

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 6x+3h-5 = 6x-5$

The limit is derivative of f(x)

b) $\lim_{x\to 0}\frac{ln(1-x)}{\sqrt{x}} = \frac{0}{0} IF$

By Hospital's rule

$\lim_{x\to 0}\frac{ln(1-x)}{\sqrt{x}} = \lim_{x\to 0}\frac{ln(1-x)'}{\sqrt{x}} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0}\frac{2\sqrt{x}}{1+x} = 0$

13. Find the distance between the point A(4, 2, -1) and the plane P whose equation is 3x-4y+z+6=0 in Euclidian space IR^3. 2 marks

Answer:

$D(P, A) = \frac{|3x4-4x2+1x(-1)+6|}{\sqrt{9+16+1}} = \frac{9}{\sqrt{26}} = \frac{9\sqrt{26}}{26}$

14. Find the sum of the first twenty terms of a geometric series if the first term is $a_1=1$ and the common ratio r=2. 2 marks

Answer:

The first term $a_1 = 2$ and common ration r =2

$S_{20} = \frac{a_1(1-r^{20})}{1-5} = \frac{1-2^{20}}{1-2} = 2^{20}-1$

15. a) Prove that $\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$ for real numbers x and y. 2 marks

b) Evaluate $\int_0^{\pi} \cos(2x)\sin(3x)dx$. 3 marks

Answer:

a) $\sin(x+y) = \cos x\sin y+\sin x\cos y$ (1)

$\sin(x-y) = \cos x\sin y-\sin x\cos y$ (2)

adding (1) and (2)

$\sin(x+y) + \sin(x-y) = 2\cos x\sin y$

since $\cos x\sin y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

b) $\int_0^{\pi} \cos(2x)\sin(3x)dx = \frac{1}{2}\int_0^{\pi}(\sin 5x + \sin(-x))dx = \frac{1}{2}\int_0^{\pi}(\sin 5x - \sin x dx)dx$

$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right]_0^{\pi} = \frac{1}{2}\left(\frac{1}{5} - 1 + \frac{1}{5} - 1\right) = -\frac{4}{5}$

SECTION B: Choose three questions in this section (45 marks each)

16. Let $f(x) = \frac{2x+8}{x+2}$ be a function with real variable x.

a) Find the domain and the intercepts with axes. 2 marks

b) Find two real numbers a and b such that $f(x) = a+\frac{b}{x+2}$. 2 marks

Write equations vertical and horizontal asymptotes. 2 marks

c) Calculate the derivative and determine its signs. 2 marks

d) Deduce variations of f (determine intervals on which f is increasing, decreasing, constant and local extrema). 2 marks

e) Sketch the graph of f showing asymptotes in the plane oxy with orthonormal system (unit = 1.5 cm). 2 marks

f) Find the area (in terms of cm^2) of the region $\begin{cases} 0 \le x \le 2 \\ 2 \le y \le f(x) \end{cases}$ 3 marks

Answer:

a) $f(x) = \frac{2x+8}{x+2}$

$Domf = \{x \in \Re : x + 2 \neq 0\} = \Re\backslash\{-2\}$

$=]-\infty, -2[\cup]-2, +\infty[$

x – intercept $\Rightarrow 2x+8 = 0 \Rightarrow x = -4$

$C \cap ox$ = point (-4, 0)

y-intercept $\Rightarrow x=0$

$\Rightarrow f(0) = 4$

$C \cap oy$ = point (0, 4)

b) $\frac{2x+8}{x+2} = a + \frac{b}{x+2} = \frac{ax+2a+b}{x+2} \Leftrightarrow \begin{cases} a = 2 \\ 2a + b = 8 \end{cases} \Leftrightarrow$ a = 2 and b = 4

Then $f(x) = 2+\frac{4}{x+2}$

$\lim_{x\to -2} f(x) = \lim_{x\to -2}(2+\frac{4}{x+2}) = \infty \Rightarrow Vertical\ asymptote\ is\ x = -2$

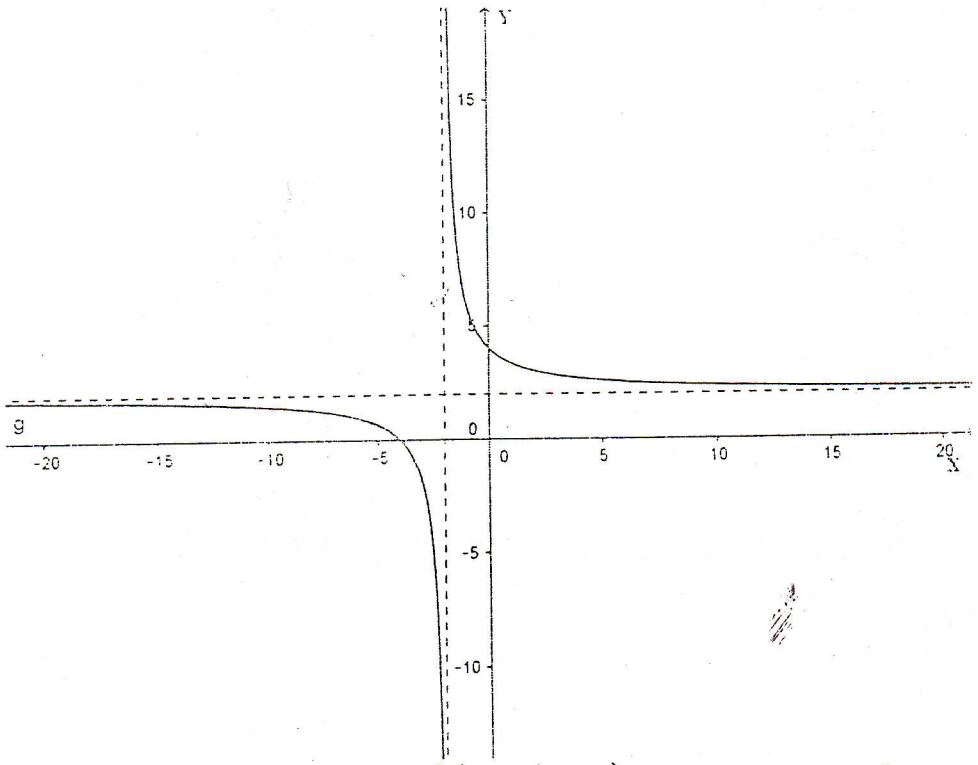
$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}(2+\frac{4}{x+2}) = 2 \Rightarrow Horizontal\ asymptote\ is\ y = 2$

c) $f'(x) = \left(2+\frac{4}{x+2}\right)' = -\frac{4}{(x+2)^2}$

$f'(x)$ is negative

d) f is always increasing on $\Re$ and no local extrema

e)



f) The area $A = \int_0^2 (f(x)-2)dx = \int_0^2 \left(2+\frac{4}{x+2}-2\right)dx = [4ln(x+2)]_0^2 = 4\ln 4 - 4\ln 2 = 4\ln 2 cm^2$

17. a) A true –false test has 20 questions. If you answer each of the questions randomly, what is the probability of answering exactly 12 questions correctly? 4 marks

b) Let $(a_n)_{a>n}$ be an arithmetic sequence with a_1=5 and a_2=5, find S_{50} (the sum of the first fifty terms). 4 marks

c) Decide whether A={(1. -3), (-2, 6)} spans or not the real vector space IR^2. 2 marks

d) Transform $9x^2+16y^2+36x-32y-92 = 0$ into a standard form of a conic. Identify the nature of the conic and graph it. 5 marks

Answer:

a) The test follows the Binomial law

$$P(x=12) = \binom{20}{12} x \left(\frac{1}{2}\right)^{12} x \left(\frac{1}{2}\right)^{8} = \binom{20}{12} x \left(\frac{1}{2}\right)^{20} = \frac{20!}{12!(8)!} x \left(\frac{1}{2}\right)^{20}$$

$$\Leftrightarrow \frac{20x19x18x17x16x15x14x13}{8x7x6x5x4x3x2x1} x \left(\frac{1}{2}\right)^{20} = 125\,970 \; x \left(\frac{1}{2}\right)^{20} = 0.12$$

b)

Let (a_n) a> n *be an arthmetic sequence*

$a_1 = 5$ and $a_2 = 5$

$d = a_2 - a_1$

$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{50}{2}[2x5 + (50-1)x0] = 250$

c) $a\begin{pmatrix}1\\-3\end{pmatrix} + b\begin{pmatrix}-2\\6\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$$\begin{array}{l|l} a - 2b = x & 3 \\ -3a + 6b = y & 1 \\ \hline 3a - 6b = 3x & \\ -3a + 6b = y & \\ \hline 3x + y = 0 & \end{array}$$

Is not spanning since $\forall \vec{x} \in IR^2$

18. a) Suppose f and g are linear transformations on real vector space IR^2 with their respective representative matrices.

$F = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ and $G = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$ relative to the standard basis.

Find the matrix that represents gof. 3 marks

b) Find a vector u such that f(u)= 2u and a vector v such that f(v) = v. 4 marks

c) Prove that B= (u, v) is a basis of the vector space IR^2. 2 marks

d) Write the matrix T that represents f relative to the basis B. 4 marks

e) Find a relationship between F and T. 2 marks

Answer:

a) $F = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ and $G = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$

The matrix that represent gof

$GoF = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix} = \begin{bmatrix}3 & -3\\-2 & 1\end{bmatrix}$

b) $F(u) = 2u \Rightarrow \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\2y\end{pmatrix}$

$\Rightarrow \begin{cases} x - y = 2x \\ 2y = 2y \end{cases}$

$\Rightarrow \begin{cases} y \in \Re \\ x = -y \end{cases}$

The vector u is in the form of (-y; y) ⇒ y(-1; 1)

$F(v) = v \Rightarrow \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$\Rightarrow \begin{cases} x - y = x \\ 2y = y \end{cases}$

$\Rightarrow \begin{cases} x \in \Re \\ y = 0 \end{cases}$

The vector v is in the form of (x, 0) ⇒ x(1, 0)

c) B = (u, v) = ((-1, 1), (1, 0)) is the basis of $\Re^2$ because $\begin{vmatrix}1 & 1\\-1 & 0\end{vmatrix} = -1 \neq 0$

d) $\begin{cases} f(u) = 2u \\ f(v) = v \end{cases} \Rightarrow \begin{cases} f(u) = 2u + 0v \\ f(v) = 0u + v \end{cases}$

Then $T = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$

e) As u = (-1, 1) = -1(1, 0) + 1(0, 1)

v = (1, 0) = 1(1, 0) + 0(0, 1)

matrix $p = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}$ is the matrix given by the canonical basis (u, v)

Then $T = p^{-1}TP$ or $P = PTP^{-1}$

19. a) Verify $\cot(\text{Arctan}(x)) = \frac{1}{x}$ for $x \neq 0$. 4 marks

b) Calculate $\lim_{x\to 0}[\cos(x)]^{\frac{1}{x^2}}$. 3 marks

c) For each complex number u=1+i and w = -1+i$\sqrt{3}$, find polar for (trigonometric form). Then find z such that $z^3 = \frac{u}{w}$. 8 marks

Answer:

a) $\cot(\text{Arc tan}(x)) = \frac{1}{x}$

Let y = Arc tan(x) $\Rightarrow x = \tan y$ with $\frac{-\pi}{2} < y < \frac{\pi}{2}$

$\Leftrightarrow \frac{1}{\tan y} = \frac{1}{x}$

$\Leftrightarrow \cot an\ y = \frac{1}{x}$

You replace the value of y by Arc tan (x)

You get $\cot(\text{Arc tan}(x)) = \frac{1}{x}$

b) $\lim_{x\to 0}(cosx)^{\frac{1}{x^2}} = 1^{\infty}$IF

Let $y = (cosx)^{\frac{1}{x^2}}$

$\ln y = \frac{1}{x^2} \ln cosx$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x^2} \ln cosx$

$= \lim_{x\to 0} \frac{\frac{-sinx}{cosx}}{2x} = \lim_{x\to 0} \frac{-sinx}{2xcosx} = \frac{0}{0}(IF)$

$= \lim_{x\to 0} \left(\frac{-cosx}{2cosx - 2xsinx}\right) = -\frac{1}{2}$

$\Rightarrow \ln \lim_{x\to 0} y = -\frac{1}{2}$

$\Rightarrow \lim_{x\to 0} y = e^{-1/2}$

$\Rightarrow \lim_{x\to 0}(cosx)^{\frac{1}{x^2}} = e^{-1/2}$

c) U = 1+i and w =-1 + i$\sqrt{3}$

$|u| = \sqrt{2}$

$u = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(cos\frac{\pi}{4} + isin\frac{\pi}{4}\right)$

$|w| = 2$

$w = 2\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= 2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$

Let $z^3 = \frac{u}{w} = \frac{\sqrt{2}}{2}\left(cos(\frac{\pi}{4} - \frac{2\pi}{3}) + isin(\frac{\pi}{4} - \frac{2\pi}{3})\right) = \frac{\sqrt{2}}{2}\left(cos - \frac{5\pi}{12} + isin - \frac{5\pi}{12}\right)$

Let z_k one of the cube roots of $\frac{u}{w}$ where k = 0.12

$z_k = (\frac{\sqrt{2}}{2})^{1/3}\left(cos\frac{1}{3}(-\frac{5\pi}{12} + 2k\pi) + isin\frac{1}{3}(-\frac{5\pi}{12} + 2k\pi)\right)$

Then

$z_0 = (\frac{\sqrt{2}}{2})^{1/3}\left(cos(-\frac{5\pi}{36}) + isin(-\frac{5\pi}{36})\right)$

$z_1 = (\frac{\sqrt{2}}{2})^{1/3}\left(cos(\frac{19\pi}{36}) + isin(\frac{19\pi}{36})\right)$

$z_2 = (\frac{\sqrt{2}}{2})^{1/3}\left(cos(\frac{43\pi}{36}) + isin(\frac{43\pi}{36})\right)$

20. Let $\begin{cases} x - 2x + z = 0 \\ 2x - y + z - 1 = 0 \end{cases}$ be the system that represents a line D of the Euclidian space IR^3.

a) Find paramagnetic equations of the plane P which is perpendicular to line D and containing the point A (-1, 2, -1). 4 marks

b) Find paramagnetic and Cartesian equations of the line E parallel to D through the point M (2, 0, 3). 4 marks

c) Find the angle between P and E. Are you surprised by the result? 3 marks

d) Calculate $\int \frac{x^3}{\sqrt{4+x^2}}dx$ by changing the variable x to u if $u^2=4+x^2$. 4 marks

Answer:

a) $D \equiv \begin{cases} x - 2x + z = 0 \\ 2x - y + z - 1 = 0 \end{cases}$

Find the direction vector of D

Solve the system

x-2y = -z

2x-y = -z+1

Let z = t

$D \equiv \begin{cases} x = \frac{2}{3} - \frac{1}{3}t \\ y = \frac{1}{3} - \frac{1}{3}t \\ z = t \end{cases}$ $t \in \Re$

The direction vector of D is $\vec{u}\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$

Since

$P \equiv -\frac{1}{3}(x+1) + \frac{1}{3}(y+1) + \frac{1}{3}(z+1) = 0$

$P \equiv x - y - 3z - 3 = 0$

b) D//E and M(2, 0, 3)

The direction vector of E is $\vec{u}\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$

Then

$E \equiv \begin{cases} x - 2 = -\frac{1}{3}t \\ y = \frac{1}{3}t \\ z - 3 = t \end{cases}$ $t \in \Re$

Eliminating t you get